Optimal Quantum Feshbach Engines

Aaron Wandhammer, Vincent Hardel, Paul-Antoine Hervieux, and Giovanni Manfredi*
University of Strasbourg, CNRS, Institut de Physique et Chimie des Matériaux de Strasbourg, UMR 7504, F-67034 Strasbourg, France

SUPPLEMENTAL MATERIAL

CONTENTS

A. Variational solution of the GPE 2

B. Nelson's stochastic quantization 3

C. Quantum-classical analogy 4

D. Cost functional 5

E. Reverse cycle 6

References 6

* giovanni.manfredi@ipcms.unistra.fr

Appendix A: Variational solution of the GPE

In order to obtain an approximate solution of the Gross-Pitaevskii equation (GPE), we first write the wavefunction in the Madelung representation $\psi(x,t) = \sqrt{P(x,t)}e^{iS(x,t)}$, where the probability density $P(x,t)$ and the phase $S(x,t)$ are real functions [1, 2]. The density obeys the continuity equation

$$\partial_t P + \frac{\hbar}{m}\partial_x \left(P\partial_x S\right) = 0, \tag{A1}$$

and the phase the quantum Hamilton-Jacobi equation

$$\hbar\partial_t S + \frac{\hbar^2}{2m}\left(\partial_x S\right)^2 + \frac{1}{2}m\omega^2 x^2 + NgP - \frac{\hbar^2}{2m}\frac{\partial_x^2\sqrt{P}}{\sqrt{P}} = 0. \tag{A2}$$

Equations (A1) and (A2) are fully equivalent to the GPE when expressed in the Madelung representation. They can also be derived from the Lagrangian density

$$\mathcal{L}[P,S] = -\hbar P\partial_t S - \frac{\hbar^2}{2m}\left[P(\partial_x S)^2 + \frac{(\partial_x P)^2}{4P}\right] - UP, \tag{A3}$$

where $U(x,t) = \frac{1}{2}(m\omega^2x^2 + NgP)$, using the standard Euler-Lagrange equations. Note that, so far, no approximations to the GPE have been made.

We now assume that the density is Gaussian and the phase is quadratic in x

$$P(x,t) = \frac{1}{\sqrt{2\pi}\sigma(t)}\, e^{-x^2/2\sigma^2(t)}, \quad S(x,t) = \alpha(t)x^2.$$

From the continuity equation (A1), it can be easily deduced that

$$\alpha(t) = \frac{m}{2\hbar}\frac{\dot{\sigma}}{\sigma}.$$

Injecting the expressions for P and S into the Lagrangian density (A3) and integrating over all space, one obtains the Lagrangian

$$L[\sigma,\ddot{\sigma}] = \int \mathcal{L}dx = -\frac{m}{2}\ddot{\sigma}\sigma - \frac{\hbar^2}{8m\sigma^2} - \frac{m\omega^2\sigma^2}{2} - \frac{Ng}{4\sqrt{\pi}\sigma}. \tag{A4}$$

The corresponding Euler-Lagrange equation

$$\frac{\partial L}{\partial\sigma} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial\ddot{\sigma}} = 0$$

yields the modified Ermakov equation

$$\ddot{\sigma} + \omega^2\sigma = \frac{\hbar^2}{4m^2\sigma^3} + \frac{Ng}{4\sqrt{\pi}m\sigma^2}. \tag{A5}$$

Plugging the solution of Eq. (A5) into the Gaussian ansatz for $\psi(x,t)$ provides the Gaussian wavefunction that best approximates the exact solution of the GPE.

Unlike similar STA approaches [3, 4], Eq. (A5) explicitly contains the interaction strength g. Hence, the approximate Gaussian solution naturally adapts itself to the different regimes of weak or strong interactions and remains reasonably accurate over a wide range of parameters, as was proven for an interacting Coulomb gas [1].

Appendix B: Nelson's stochastic quantization

In a seminal work [5], Nelson showed that the Schrödinger equation can be recast as the first-order stochastic process (see also [6])

$$\mathrm{d}x(t) = b(x(t),t)\mathrm{d}t + \mathrm{d}W. \tag{B1}$$

The deterministic drift velocity of the process can be written as $b(x,t) = v(x,t) + u(x,t)$, where $v = \hbar\partial_x S/m$ is the *current* velocity (as in Bohmian mechanics) and $u = D\,\partial_x \ln P(x,t)$ is the density-dependent *osmotic* velocity.

Nelson's derivation starts by assuming that particle trajectories are diffusive processes with forward and backward velocities $b_\pm$ and Wiener processes $\mathrm{d}W_\pm$

$$\mathrm{d}x(t) = b_\pm(x(t),t)\mathrm{d}t + \mathrm{d}W_\pm, \tag{B2}$$

The symmetric and antisymmetric combinations define the current and osmotic velocities, $v = (b_+ + b_-)/2$ and $u = (b_+ - b_-)/2$. The requirement that the forward and backward Fokker–Planck equations associated to Eq. (B2) describe the same probability density P, yields $u = D\,\partial_x \ln P$. This guarantees that the total process, despite being stochastic, preserves time-reversal properties. Imposing Newton's second law requires that v be the gradient of a function, $v = \hbar\partial_x S/m$, with S satisfying the quantum Hamilton–Jacobi equations (A2). Further assuming $D = \hbar/(2m)$, the equations for P and S combine into the time-dependent Schrödinger equation for the wavefunction $\Psi = \sqrt{P}\,e^{iS}$.

A simpler way to obtain Nelson's stochastic process is to consider the continuity equation (A1), which we rewrite as $\partial_t P + \partial_x(vP) = 0$. By adding the quantity $\partial_x[P\,D\partial_x(\ln P)] = D\partial_x^2 P$ on both sides of the equation, yields the Fokker–Planck equation:

$$\partial_t P + \partial_x[(v+u)P] = D\,\partial_x^2 P, \tag{B3}$$

which is equivalent to the Nelson stochastic process defined by Eq. (B1). The evolution of the phase S, which defines the current velocity v, is still given by the Hamilton–Jacobi equation (A2). Therefore, combined together, Eq. (B1) [or (B3)] and Eq. (A2) are equivalent to the Schrödinger dynamics for the wavefunction $\Psi = \sqrt{P}\,e^{iS}$.

Furthermore, the above derivation highlights the fact that the Nelson approach also holds for a nonlinear Schrödinger equation such as the GPE. It suffices to employ the relevant Hamilton–Jacobi equation (A2) that contains the self-interaction term NgP.

Nelson's dynamics has been the object of some debate about its ability to reproduce the entirety of quantum mechanical predictions [7, 8]. We cannot address this issue in detail here, but note that the debate has primarily concerned position correlations, such as $\langle x(t)\,x(t')\rangle$, whereas it is well established that Nelson's dynamics reproduces the correct probability density $P(x,t)$. We further emphasize that Nelson's approach has been successfully used to investigate the Born rule [9] and to derive quantum optimal protocols [10], yielding results in full agreement with the predictions of quantum mechanics.

Appendix C: Quantum-classical analogy

Equating the deterministic term of the classical Ornstein-Uhlenbeck process [Eq. (4) of the main text]

$$\mathrm{d}x = -\gamma^{-1}\bar{\kappa}(t)\, x\, \mathrm{d}t + \mathrm{d}W$$

to the same term in Nelson's equation [Eq. (3) of the main text]

$$\mathrm{d}x = (\hbar/2m)\left[4\alpha(t) - \sigma^{-2}(t)\right] x\, \mathrm{d}t + \mathrm{d}W,$$

we get

$$-\frac{\bar{\kappa}}{\gamma} = \frac{\hbar}{m}\left(2\alpha - \frac{1}{2\sigma^2}\right),$$

whose time derivative yields

$$\dot{\bar{\kappa}} = -\frac{\hbar\gamma}{m}\left(\frac{m}{\hbar}\frac{\ddot{\sigma}\sigma - \dot{\sigma}^2}{\sigma^2} + \frac{\dot{\sigma}}{\sigma^3}\right).$$

Solving for $\ddot{\sigma}$ and substituting into Eq. (A5), we obtain

$$\kappa - \frac{m\dot{\bar{\kappa}}}{\gamma} - \hbar\frac{\dot{\sigma}}{\sigma^3} + \frac{m\dot{\sigma}^2}{\sigma^2} = \frac{\hbar^2}{4m\sigma^4} + \frac{Ng}{4\sqrt{\pi}\sigma^3},$$

where we used $\kappa = m\omega^2$. Remembering that $\sigma^2 = s$ and using

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{2}{\gamma}\left[D\gamma - \bar{\kappa}(s)\, s\right], \tag{C1}$$

[Eq. (6) of the main text] to express $\dot{\sigma} = (D\gamma - \bar{\kappa}\sigma^2)/(\gamma\sigma)$, one finally obtains Eq. (5) of the main text.

Appendix D: Cost functional

We want to minimize the total cost functional $J[\bar{\kappa}, \bar{\kappa}']$ defined in Eq. (8) of the main text. This functional is composed of three terms: the first term represents the transition time, while the last term is a regularizing term that prevents discontinuities in the protocol. The middle term is an arbitrary cost term representing the (yet unspecified) physical quantity that we want to minimize. The total cost functional can be written as

$$J[\bar{\kappa}, \bar{\kappa}'] = \int_{s_i}^{s_f} L(s, \bar{\kappa}, \bar{\kappa}')\, \mathrm{d}s,$$

where the Lagrangian L is

$$L[s, \bar{\kappa}, \bar{\kappa}'] = \frac{1}{2}\frac{\gamma}{D\gamma - s\bar{\kappa}(s)} + \lambda f(s, \bar{\kappa}, \bar{\kappa}') + \mu|\bar{\kappa}'|^2.$$

The associated Euler-Lagrange equation

$$\frac{\partial L}{\partial \bar{\kappa}} - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial \bar{\kappa}'} = 0$$

yields

$$2\mu \frac{\mathrm{d}^2\bar{\kappa}}{\mathrm{d}s^2} = \frac{\gamma s}{(D\gamma - s\bar{\kappa})^2} + \lambda\frac{\partial f}{\partial \bar{\kappa}} - \lambda\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial f}{\partial \bar{\kappa}'}. \tag{D1}$$

Equation (D1) can be solved numerically for $\bar{\kappa}(s)$ as a boundary-value problem on the interval $[s_i, s_f]$, with boundary conditions $\bar{\kappa}(s_{i,f}) = D\gamma/s_{i,f}$ corresponding to steady-state solutions of Eq. (C1).

From the solution $\bar{\kappa}(s)$ in terms of the variance s, one obtains $\bar{\kappa}(t) \equiv \bar{\kappa}(s(t))$ as a function of time, by using the transformation

$$t(s) = \int_{s_i}^{s} \frac{\mathrm{d}s}{\mathrm{d}s/\mathrm{d}t} = \int_{s_i}^{s} \frac{\gamma}{2}\frac{\mathrm{d}s}{D\gamma - \bar{\kappa}(s)}.$$

The corresponding control parameters $\kappa(t)$ or $g(t)$ then follow directly from Eq. (5) of the main text.

In the present case, we want to minimize the squared phase of the wavefunction $\alpha(t) = \frac{m}{2\hbar}\frac{\dot{\sigma}}{\sigma} = \frac{m}{4\hbar}\frac{\dot{s}}{s}$, i.e. the integral $\int_0^{t_f} \alpha^2(t)\, \mathrm{d}t$. This amounts to requiring that the evolution remains as close as possible to an adiabatic one [10]. The second term of the cost functional $J[\bar{\kappa}, \bar{\kappa}']$ is then

$$\begin{aligned}
\int_0^{t_f} \alpha^2(t)\, \mathrm{d}t &= \int_0^{t_f} \left(\frac{m}{4\hbar}\frac{\dot{s}}{s}\right)^2 \mathrm{d}t \\
&= \int_{s_i}^{s_f} \left(\frac{m}{4\hbar s}\frac{2\,(D\gamma - s\bar{\kappa})}{\gamma}\right)^2 \frac{\gamma}{2\,(D\gamma - s\bar{\kappa})}\, \mathrm{d}s \\
&= \frac{m^2}{8\gamma\hbar^2}\int_{s_i}^{s_f} \frac{D\gamma - s\bar{\kappa}}{s^2}\, \mathrm{d}s,
\end{aligned}$$

where we have used Eq. (C1). From this, we immediately obtain the function

$$f(s, \bar{\kappa}, \bar{\kappa}') = \frac{m^2}{8\gamma\hbar^2}\frac{D\gamma - s\bar{\kappa}}{s^2}. \tag{D2}$$

Injecting the Eq. (D2) into Eq. (D1), we finally obtain Eq. (9) of the main text.

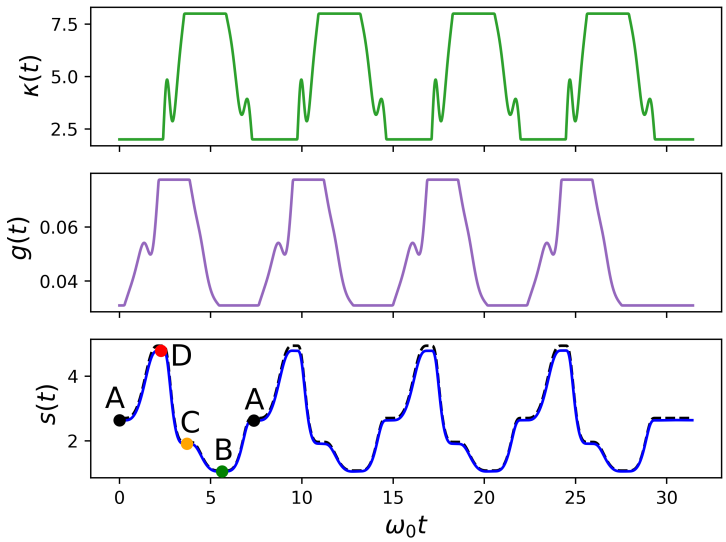


Figure 1: Same as Fig. 3 of the main text for the reverse cycle ADCBA.

Appendix E: Reverse cycle

In Fig. 1, we show the reverse cycle ADCBA corresponding to the direct cycle displayed in Fig. 3 of the main text. The reverse cycle is as stable as the direct one.

[1] F. Haas, G. Manfredi, P. K. Shukla, and P.-A. Hervieux, Breather mode in the many-electron dynamics of semiconductor quantum wells, Phys. Rev. B **80**, 073301 (2009).

[2] J. Hurst, K. Lévêque-Simon, P.-A. Hervieux, G. Manfredi, and F. Haas, High-harmonic generation in a quantum electron gas trapped in a nonparabolic and anisotropic well, Phys. Rev. B **93**, 205402 (2016).

[3] J.-F. Schaff, X.-L. Song, P. Capuzzi, P. Vignolo, and G. Labeyrie, Shortcut to adiabaticity for an interacting Bose-Einstein condensate, Europhysics Letters **93**, 23001 (2011).

[4] C. Mishra, T. Busch, and T. Fogarty, Shortcuts to adiabaticity in anisotropic Bose-Einstein condensates, New Journal of Physics **27**, 033009 (2025).

[5] E. Nelson, Derivation of the Schrödinger Equation from Newtonian Mechanics, Physical Review **150**, 1079 (1966).

[6] G. Bacciagaluppi, Nelsonian mechanics revisited, Foundations of Physics Letters **12**, 1 (1999).

[7] H. Grabert, P. Hänggi, and P. Talkner, Is quantum mechanics equivalent to a classical stochastic process?, Phys. Rev. A **19**, 2440 (1979).

[8] P. Blanchard, S. Golin, and M. Serva, Repeated measurements in stochastic mechanics, Phys. Rev. D **34**, 3732 (1986).

[9] V. Hardel, P.-A. Hervieux, and G. Manfredi, Relaxation to quantum equilibrium and the Born rule in Nelson's stochastic dynamics, Found. Phys. **53**, 89 (2023).

[10] V. Hardel, G. Manfredi, P.-A. Hervieux, and R. Goerlich, Shortcuts to adiabaticity in harmonic traps: A quantum-classical analog, Physical Review E **110**, 054138 (2024).